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Mechanical Impedance and Mobility Useful but Unused

Technical Paper 210

By Dale Burger

MECHANICAL IMPEDANCE AND MOBILITY, USEFUL BUT UNUSED

INTRODUCTION

This paper discusses mechanical impedance and mobility concepts. It points out the definite advantage of using mobility instead of mechanical impedance for mechanical systems. Mobility retains the essential character of a mechanical system, parallel systems remain parallel, etc., even after the transformation to standard "lumped" values. The lumped value system may be solved by electrical engineering mathematical methods. Measurements are simply made with dynamic force gages, accelerometers and phase meters.

The concept of mobility makes possible the straightforward analysis of complex structures under dynamic loading. The measurement of dynamic forces and motions is analogous to that of electronic impedance measurement. Acceleration measured at a point in a structure and compared to the driving force and frequency will indicate the lumped constants of Mass, Spring and Damping. Further analysis makes possible the design of mountings for idealized impedance matching or mismatching to the supporting structure, whose characteristics have been measured in the same manner. Furthermore, the knowledge of the lumped dynamic constants for any complex structure at given frequencies makes it possible to predict accurately the response of that system under varying conditions. One application of this concept is the measurement of dynamic forces in large rotating equipment so as to easily determine proper design criteria.

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Four steps are presented for effective utilization of the analogy method of vibration analysis. These steps are: (1) selection of the proper set of equations; (2) transformation of an actual system to a lumped value system; (3) measurement of the system constants over the applicable frequency range; (4) solution of equations. The problems present in the transformation of a simple distributed system are illustrated by an example. Equations for electrical, mechanical, acoustical and hydraulic systems are presented. A complete derivation of mechanical mobility is presented along with a problem in this field.

Most engineers are familiar with the concept of applying solutions of one field to the problems of another by use of analogous transformations. Unfortunately, this powerful technique has not been used to any great extent on the complex problems of present day vibration analysis. Some reason for this lack of application can be found in the system transformation that has been taught in colleges, which is not an easily visualized one. It has also been difficult to check if the schematic representation of any system is a correct one. This paper presents a simple method of system transformations and also includes a glimpse of new aids to vibration analysis due to recent advances in instrumentation.

If the differential equation of one system is identical in form to the differential equation of any other system, then the solution of the first system equation may be used as the solution of the second system with just a change of constants. A change from one system to another with the same equation form is a system transformation. In practice, there exists two types of transformations—one is a dual transformation and the other is an analogous transformation. A simple dual transformation within a mechanical system is illustrated in Figure 1 and a similar transformation in an electrical system is shown in Figure 2. The best use of the dual transformation is to change one type of generator to a more familiar or easily workable one, such as the current generator to voltage generator change in Figure 2. An analogous transformation from one system to another must preserve the visual similarity of system schematics. This is the correspondence between Figures 1a and 2a or 1b and 2b.

The classical textbook analogy, which is a combined analogous and dual transformation, is between Figures 1a and 2b or 1b and 2a. This confusing practice came about due to the mathematically indefensible belief that an electromotive force should be the analog of a mechanical force. An analogous transformation is much superior to a dual transformation,

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both from the standpoint of easy visualization and the mathematical treatment of the boundary conditions of the original system.

A basic understanding of system transformations may be obtained by considering the action of system variables at a junction within the system. In an ideal mechanical system this junction is a point from which three or more massless, infinitely rigid bars radiate; in an electrical system it is a node or junction of three or more ideal conductors; and in a hydraulic or acoustic system it is a chamber with three or more tubes leading from it. Velocity in a mechanical system can be called an "across" variable since the velocity of a bar from one side of the junction must be the same as a bar from the other side of the junction as long as the velocities are measured in the same direction. Voltage in the electrical system must always be the same in any conductor running from the junction. A hydraulic or acoustic system has equal pressures in tubes from the junction. The other variable in each of the above systems will change at the junction and can be called the "through" variable. "Through" variables in the various systems are: force in mechanics; current in electrical circuits; and flow rate in hydraulics or acoustics. Listed below are the differential equations for the various systems with the "across" variable equation given first in each set.

1. ELECTRICAL

A. $L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = v\ (t)$

B. $C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int vdt = i\ (t)$

2. MECHANICAL

A. $\frac{1}{K}\frac{df}{dt} + \frac{1}{D}f + \frac{1}{M}\int fdt = v\ (t)$

B. $M\frac{dv}{dt} + Dv + K\int vdt = f\ (t)$

3. HYDRAULIC

A. $M\frac{dv}{dt} + R_H v + \frac{1}{C_H}\int vdt = p\ (t)$

B. $C\frac{dp}{dt} + \frac{1}{R_H}p + \frac{1}{M}\int pdt = v\ (t)$

4. ACOUSTIC

A. $M\frac{dX}{dt} + R_A X + \frac{1}{C_A}\int Xdt = P\ (t)$

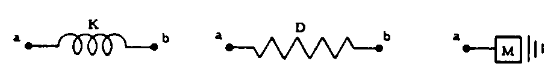
B. $C_A\frac{dP}{dt} + \frac{1}{R_A}P + \frac{1}{M}\int Pdt = X\ (t)$

Now that the system variables and equations have been considered, it is necessary to inspect the system references. The reference for mechanical systems is the ground which has infinite impedance and mass. Hydraulic and acoustic systems have the open sky as reference which has zero impedance and infinite compliance. This difference in primary references gives rise to two different ways of visualizing the system schematics. One way is to use the admittance of each element as referred to ground and the other is to use the impedance of an element as referred to ground. Since mechanical ground has an infinite impedance, it must have a zero admittance. Unfortunately, there is a problem in language at this point. Mechanical admittance is historically defined as $Y_m = v$ (through)/f (across). This is not what was shown to be true of a mechanical system where velocity is an across variable and force is a through variable. The reason for this lack of correspondence is that mechanical admittance really belongs in a hydraulic system while a new term, mechanical mobility, which was coined by F. A. Firestone and is defined as $z = v$ (across)/f (through), is the correct method of analysis for a mechanical system. Hydraulic or acoustic systems have zero impedance references, such as the sky or a lake at zero elevation, so these systems present no ingrained difficulties in schematic representation using electrical impedances. Figure 3 shows a mechanical system and an acoustical system, each with an "across" variable generator and each transformed to an electrical schematic. The dual systems for those shown in Figure 3 are given in Figure 4. It should be noted here that an analogous transformation must be made with care since system non-linearities may not be recognized and included in the transformation.

Mechanical mobility is not a new development since it was introduced in 1933, but it has not received much notice until recently. As stated earlier, the basic definition of mechanical mobility is: $z = v$ (across)/f (through), where v is analogous to voltage and f is analogous to current. Since most mechanical systems are analyzed by using a force generator, they tend to have mechanical schematics like that in Figure 5. Equations 1A and 1B are the analogous equations for this problem and the schematic values in Figure 6 are derived from these equations. The actual mobilities of single elements are derived below.

Basic Symbols:

Spring Damper or friction Mass



Note: A mass only has one terminal since the inertia force is absolute with respect to ground.

Mobility of a spring:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{f}{k}\right) = \frac{i\omega f_0 e^{i\omega}}{k} \text{ ; when } f = f_0 e^{i\omega}$$

$$z = \frac{v}{f} = \frac{i\omega f_0 e^{i\omega}}{k} \times \frac{1}{f_0 e^{i\omega}} = \frac{i\omega}{k}$$

Mobility of a damper:

$$z = \frac{v}{f} = \frac{f}{D} \times \frac{1}{f} = \frac{1}{D}, \text{ since } f = Dv$$

Mobility of a mass:

$$f = M\frac{dv}{dt}, \therefore v = \int \frac{f}{M} dt = \frac{f_0 e^{i\omega}}{i\omega M}, \text{ when } f = f_0 e^{i\omega}$$

$$z = \frac{f_0 e^{i\omega}}{i\omega M} \times \frac{1}{f_0 e^{i\omega}} = \frac{1}{i\omega M}$$

A direct system solution is possible without a transformation. The mechanical schematic of Figure 5b can be converted into algebraic equations quite rapidly by use of the mobilities of its elements. The mobility of point A is:

$$z_A = \frac{1}{\frac{1}{z_{D_1}} + \frac{1}{z_{K_1}} + \frac{1}{z_{M_1}}} = \frac{1}{\frac{1}{\frac{1}{D_1}} + \frac{1}{\frac{i\omega}{k_1}} + \frac{1}{\frac{1}{i\omega M_1}}}$$

The mobility of point B is:

$$z_B = \frac{1}{\frac{1}{z_{D_2}} + \frac{1}{z_{M_2}} + \frac{1}{z_{K_2} + z_A}}$$

The above mobility development and problem solution is only valid for systems where the driving force varies sinusoidally. However, by application of a Fouier series this restriction is nearly removed. The use of an imaginary coordinate was deliberate since the imaginary part represents recoverable power and the real part represents dissipated power.

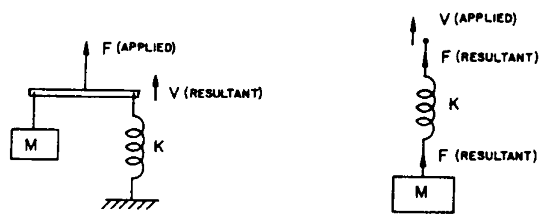
All of the preceding material may be found in one or more reference books and this information allows an engineer to make transformations in simple systems. There are four steps necessary to effectively utilize the analogy method of analysis. These steps are: (1) selection of the proper system equations, (2) transformation of the actual system to a lumped value system; (3) measurement of the system constants; (4) solution of the equations. The first step of selecting the proper system equations has been covered and is not too difficult once the underlying principles are understood. Step number two is one which is very conveniently glossed over in every book in the field. At present the general method of transformation consists of drawing a system schematic using past experience as a guide and then measuring static system values. This kind of approach may eventually lead to correct answers after numerous tries but it certainly is not good enough for use on a complex problem. An obvious need is some means of taking the guesswork out of system transformation and the measurement of system constants.

Instrumentation has been developed to accurately measure the magnitude of mechanical mobility or impedance and also their phase angles with respect to the driving frequency. After plots have been made of these quantities versus forcing frequency, then an electrical schematic can be derived by curve matching. This method leaves something to be desired since it may be difficult to find the electrical system which will satisfy the plotted curve. The only way around this problem is the use of an approximate mechanical schematic as a guide. Usually enough information may be derived from the mechanical schematic to obtain the correct electrical schematic. The best feature of this method is its complete generality and flexibility of application since a prototype model or a finished unit may be measured regardless of complexity. With really complex systems it may even be possible to let a computing machine do the curve matching. Step number three, the measurement of system constants, has already been discussed and an adequate solution to the problem is available. The last step of problem solution is fairly straightforward unless the mathematics gets a little difficult due to non-sinusoidal input conditions or non-linear system characteristics. This is the place where advanced electric circuit theory is very useful. La Place transforms, Heaviside operational calculus and switching circuit transient solutions are just some of the powerful tools available in the electrical field.

A great deal of effort has been expended in obtaining a solution technique for vibration problems. The question now is what can be accomplished with the technique. If an undesirable system resonance or antiresonance must be reduced or removed, it is difficult to decide which part of the system must be modified so that no desirable characteristics of the system are lost. There are many techniques in electrical circuit theory which show the effect on a system of the variation of single circuit elements. Another method of attack is to filter out the undesirable frequency range with an electrical filter and then transform the filter to its mechanical or hydraulic equivalent. Designers can look up or develop by breadboards an electrical circuit which has any desired frequency response. A simple transformation to the proper system and a whole program of expensive model building is rendered unnecessary. One of the most difficult problems at present is that of high frequency mechanical, acoustical or hydraulic systems. All of these systems have to be handled as distributed systems instead of lumped systems even at fairly moderate frequencies. Here again, the work done on distributed electrical systems such as transmission lines or high frequency antennas can be of enormous value to the designer. Figure 7a shows a simple mechanical system, Figure 7b is the usual low frequency schematic and Figure 7c illustrates some of the modifications that may be necessary at higher operating frequencies. In Figure 7b the actual system spring of 7a can be shown as an idealized spring since its mass is small and the mass of the actual system is not too large so it can be considered a mobile element. When the forcing frequency rises the mobility equations that were derived earlier show what will happen. First, $z_k = \frac{i\omega}{k}$ indicates that the spring will become very compliant. Second, $z_m = \frac{i}{i\omega m}$ indicates that the large mass will look like ground and the previously negligible mass of the spring must be considered. The inherent damping of the spring material will also be of interest. Figure 7c is the schematic of the high frequency system.

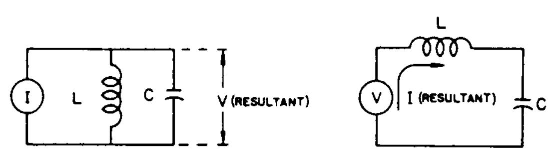
One possibility of systems has not been mentioned so far and this is the interaction of two or more systems in the same design. Multiple systems are an everyday design problem since a simple product like a loudspeaker is really an electro-mechanical-acoustical system. Obviously there must be some provision made for coupling these different systems. The theory of system coupling is fairly recent and F. A. Firestone has published an excellent article on system "meshers" in "The Journal of the Acoustical Society of America." If space permitted, it would be very illuminating to show these "meshers" and some of the other highly technical aspects of system transformations.

This paper was written to acquaint engineers with recent advances in the field of analogous transformations as they might be applied to present design problems. The more difficult mathematical treatments and some necessary but highly theoretical concepts have not been included for lack of space, but all of this material can be found in the references. If the concept of dual and analogous transformations has been clarified or the information on mobility and new problem approaches has aroused interest, then this paper has accomplished its purpose.



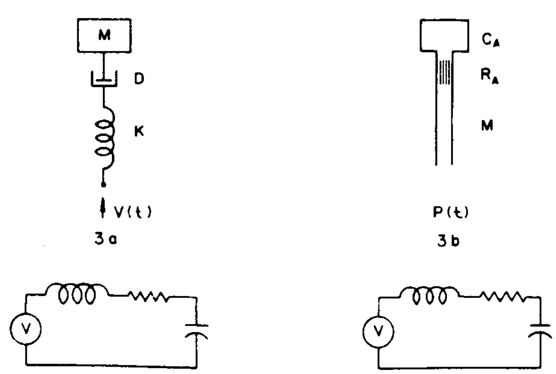
1a 1b

FIGURE 1



2a 2b

FIGURE 2



3c 3d

FIGURE 3

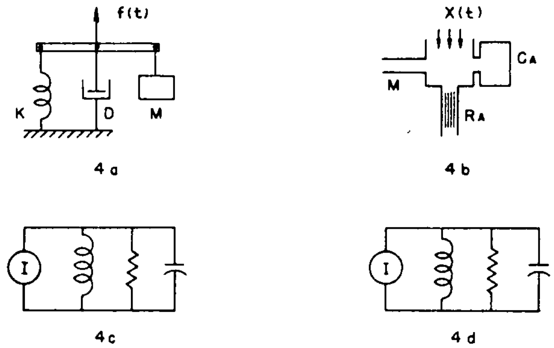


FIGURE 4

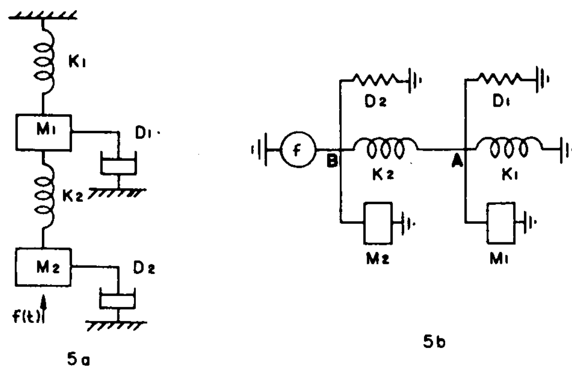


FIGURE 5

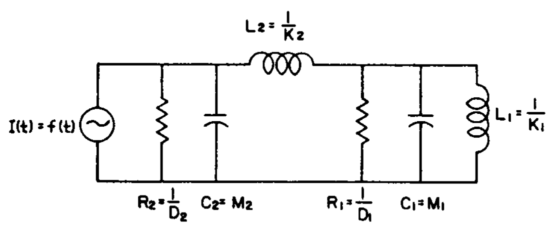


FIGURE 6

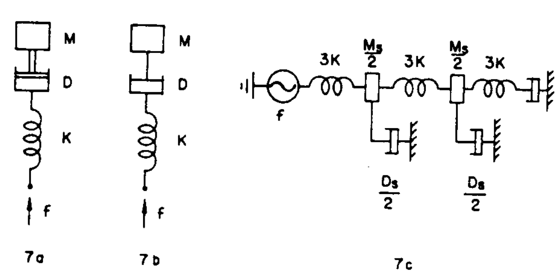


FIGURE 7

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